

FORTY-PLUS ANNOTATED QUESTIONS ABOUT LARGE TOPOLOGICAL GROUPS

VLADIMIR PESTOV

ABSTRACT. This is a selection of open problems dealing with “large” (non locally compact) topological groups and concerning extreme amenability (fixed point on compacta property), oscillation stability, universal minimal flows and other aspects of universality, and unitary representations.

A topological group G is *extremely amenable*, or has the *fixed point on compacta property*, if every continuous action of G on a compact Hausdorff space has a G -fixed point. Here are some important examples of such groups.

Example 1. The unitary group $U(\ell^2)$ of the separable Hilbert space ℓ^2 with the strong operator topology (that is, the topology of pointwise convergence on ℓ^2) (Gromov and Milman [31]).

Example 2. The group $L^1((0, 1), \mathbb{T})$ of all equivalence classes of Borel maps from the unit interval to the circle with the L^1 -metric $d(f, g) = \int_0^1 |f(x) - g(x)| dx$ (Glasner [24], Furstenberg and Weiss, unpublished).

Example 3. The group $\text{Aut}(\mathbb{Q}, \leq)$ of all order-preserving bijections of the rationals, equipped with the natural Polish group topology of pointwise convergence on \mathbb{Q} considered as a discrete space and, as an immediate corollary, the group $\text{Homeo}_+[0, 1]$ of all homeomorphisms of the closed unit interval, preserving the endpoints, equipped with the compact-open topology (the present author [42]).

The above property is not uncommon among concrete “large” topological groups coming from diverse parts of mathematics. In addition to the above quoted articles, we recommend [23, 37] and the book [45].

The group in example 2 is monothetic, that is, contains a dense subgroup isomorphic to the additive group of integers \mathbb{Z} . Notice that every abelian extremely amenable group G is *minimally almost periodic*, that is, admits no non-trivial continuous characters (the book [14] is a useful reference): indeed, if $\chi: G \rightarrow \mathbb{T}$ is such a character, then $(g, z) \mapsto \chi(g)z$ defines a continuous action of G on \mathbb{T} without fixed points. The converse remains open.

Question 1 (Eli Glasner [24]). *Does there exist a monothetic topological group that is minimally almost periodic but not extremely amenable?* 1001 ?

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An equivalent question is: does there exist a topology on the group \mathbb{Z} of integers making it into a topological group that admits a free action on a compact space but has no non-trivial characters?

Suppose the answer to the above question is in the positive, and let τ be a minimally almost periodic Hausdorff group topology on \mathbb{Z} admitting a free continuous action on a compact space X . Let $x_0 \in X$. Find an open neighbourhood V of x_0 with $1 \cdot V \cap V = \emptyset$. It is not difficult to verify that the set $S = \{n \in \mathbb{Z} : nx_0 \in V\}$ is *relatively dense* in \mathbb{Z} , that is, the size of gaps between two subsequent elements of S is uniformly bounded from above, and at the same time, the closure of $S - S = \{n - m : n, m \in S\}$ is a proper subset of \mathbb{Z} . The interior of $S - S$ in the Bohr topology on \mathbb{Z} (the finest precompact group topology) is therefore not everywhere dense in (\mathbb{Z}, τ) . Assuming this interior is non-empty, one can now verify that the τ -closures of elements of the Bohr topology on \mathbb{Z} form a base for a precompact group topology that is nontrivial and coarser than τ , contradicting the assumed minimal almost periodicity of (\mathbb{Z}, τ) . Thus, a positive answer to Glasner's question would answer in the negative the following very old question from combinatorial number theory/harmonic analysis, rooted in the classical work of Bogoliuboff, Følner [18], Cotlar and Ricabarra [11], Veech [62], and Ellis and Keynes [16]:

? 1002 Question 2. *Let S be a relatively dense subset of the integers. Is the set $S - S$ a Bohr neighbourhood of zero in \mathbb{Z} ?*

We refer the reader to Glasner's original work [24] for more on the above. See also [64, 43, 45].

? 1003 Question 3. *Does there exist an abelian minimally almost periodic topological group acting freely on a compact space?*

This does not seem to be equivalent to Glasner's problem, because there are examples of minimally almost periodic abelian Polish groups whose every monothetic subgroup is discrete, such as $L^p(0, 1)$ with $0 < p < 1$.

There are numerous known ways to construct monothetic minimally almost periodic groups [1, 14, 4, 48]. The problem is verifying their (non) extreme amenability. The most general result presently known asserting non extreme amenability of a topological group is:

Theorem (Veech [61]). *Every locally compact group admits a free action on a compact space.*

Since every locally compact abelian group admits sufficiently many characters, one cannot employ Veech theorem to answer Glasner's question. Can the result be extended? Recall that a topological space X is called a k_ω -space (or: a *hemi-compact* space) if it admits a countable cover K_n , $n \in \mathbb{N}$ by compact subsets in such a way that an $A \subseteq X$ is closed if and only if $A \cap K_n$ is closed for all n . For example, every countable CW -complex, every second countable locally compact space, and the free topological group [28] on a compact space are such.

? 1004 Question 4. *Is it true that every topological group G that is a k_ω -space admits a free action on a compact space?*

Question 5. *Same, for abelian topological groups that are k_ω -spaces.*

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A positive answer would have answered in the affirmative Glasner's question because there are examples of minimally almost periodic k_ω group topologies on the group \mathbb{Z} of integers [48].

Recall that the *Urysohn universal metric space* \mathbb{U} is the (unique up to an isometry) complete separable metric space that is *ultrahomogeneous* (every isometry between two finite subsets extends to a global self-isometry of \mathbb{U}) and *universal* (\mathbb{U} contains an isometric copy of every separable metric space) [54, 63, 30, 21]. The group $\text{Iso}(\mathbb{U})$ of all self-isometries of \mathbb{U} , equipped with the topology of pointwise convergence (which coincides with the compact-open topology), is a Polish topological group with a number of remarkable properties. In particular, $\text{Iso}(\mathbb{U})$ is a universal second-countable topological group [57, 58] and is extremely amenable [44].

Question 6. *Is the group $\text{Iso}(\mathbb{U})$ divisible, that is, does every element possess roots of every positive natural order?*¹

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Returning to Glasner's question 1, every element f of $\text{Iso}(\mathbb{U})$ generates a monothetic Polish subgroup, so one can talk of *generic monothetic subgroups* of $\text{Iso}(\mathbb{U})$ (in the sense of Baire category).

Question 7 (Glasner and Pestov, 2001, unpublished). *Is a generic monothetic subgroup of the isometry group $\text{Iso}(\mathbb{U})$ of the Urysohn metric space minimally almost periodic?*

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Question 8 (Glasner and Pestov). *Is a generic monothetic subgroup of $\text{Iso}(\mathbb{U})$ of the Urysohn metric space extremely amenable?*

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The concept of the universal Urysohn metric space admits numerous modifications. For instance, one can study the universal Urysohn metric space \mathbb{U}_1 of diameter one (it is isometric to every sphere of radius $1/2$ in \mathbb{U}). By analogy with the unitary group $U(\ell^2)$, it is natural to consider the *uniform topology* on the isometry group $\text{Iso}(\mathbb{U}_1)$, given by the bi-invariant uniform metric $d(f, g) = \sup_{x \in \mathbb{U}_1} d_{\mathbb{U}_1}(f(x), g(x))$. It is strictly finer than the strong topology.

Question 9. *Is the uniform topology on $\text{Iso}(\mathbb{U}_1)$ non-discrete?*²

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Question 10. *Does $\text{Iso}(\mathbb{U}_1)$ possess a uniform neighbourhood of zero covered by one-parameter subgroups?*

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Question 11. *Does $\text{Iso}(\mathbb{U}_1)$ have a uniform neighbourhood of zero not containing non-trivial subgroups?*

1011 ?

Question 12. *Is $\text{Iso}(\mathbb{U}_1)$ with the uniform topology a Banach–Lie group?*

1012 ?

¹Recently Julien Melleray has announced a negative answer (private communication).

²According to Julien Melleray (a private communication), the answer is yes.

The authors of [50] have established the following result as an application of a new automatic continuity-type theorem and Example 3 above.

Theorem (Rosendal and Solecki [50]). *The group $\text{Aut}(\mathbb{Q}, \leq)$, considered as a discrete group, has the fixed point on metric compacta property, that is, every action of $\text{Aut}(\mathbb{Q}, \leq)$ on a compact metric space by homeomorphisms has a common fixed point. The same is true of the group $\text{Homeo}_+[0, 1]$.*

This is particularly surprising in view of the Veech theorem, or, rather, its earlier version established by Ellis [15]: every discrete group G acts freely on a suitable compact space by homeomorphisms (e.g. on βG). The two results seem to nearly contradict each other!

? 1013 **Question 13.** *Does the unitary group $U(\ell^2)$, viewed as a discrete group, have the fixed point on metric compacta property?*

? 1014 **Question 14.** *The same question for the isometry group of the Urysohn space \mathbb{U}_1 of diameter one.*

Extreme amenability is a strong form of *amenability*, an important classical property of topological groups. A topological group G is *amenable* if every compact G -space admits an invariant probability Borel measure. Another reformulation: the space $\text{RUCB}(G)$ of all bounded right uniformly continuous real-valued functions on G admits a left-invariant mean, that is, a positive functional ϕ of norm 1 and the property $\phi({}^g f) = \phi(f)$ for all $g \in G$, $f \in \text{RUCB}(G)$, where ${}^g f(x) = f(g^{-1}x)$. (Recall that the *right uniform structure* on G is generated by entourages of the diagonal of the form $V_R = \{(x, y) \in G \times G : xy^{-1} \in V\}$, where V is a neighbourhood of identity. For the *left* uniformity, the formula becomes $x^{-1}y \in V$.) For a general reference to amenability, see e.g. [41].

? 1015 **Question 15** (A. Carey and H. Grundling [9]). *Let X be a smooth compact manifold, and let G be a compact (simple) Lie group. Is the group $C^\infty(X, G)$ of all smooth maps from X to G , equipped with the pointwise operations and the C^∞ topology, amenable?*

This question is motivated by gauge theory models of mathematical physics [9].

? 1016 **Question 16.** *To begin with, is the group of all continuous maps $C([0, 1], SO(3))$ with the topology of uniform convergence amenable?*

The following way to prove extreme amenability of topological groups was developed by Gromov and Milman [31]. A topological group G is called a *Lévy group* if there exists an increasing net (K_α) of compact subgroups whose union is everywhere dense in G , having the following property. Let μ_α denote the Haar measure on the group K_α , normalized to one ($\mu_\alpha(K_\alpha) = 1$). If $A \subseteq G$ is a Borel subset such that $\liminf_\alpha \mu_\alpha(A \cap K_\alpha) > 0$, then for every neighbourhood V of identity in G one has $\lim_\alpha \mu_\alpha(VA \cap K_\alpha) = 1$. (Such a family of compact subgroups is called a *Lévy family*.)

Theorem (Gromov and Milman [31]). *Every Lévy group is extremely amenable.*

Proof. We will give a proof in the case of a second-countable G , where one can assume the net (K_α) to be an increasing sequence. For every free ultrafilter ξ on \mathbb{N} the formula $\mu(A) = \lim_{n \rightarrow \xi} \mu_n(A \cap K_n)$ defines a *finitely-additive* measure on G of total mass one, invariant under multiplication on the left by elements of the everywhere dense subgroup $\underline{G} = \cup_{n=1}^\infty K_n$. Besides, μ has the property that if $\mu(A) > 0$, then for every non-empty open V one has $\mu(VA) = 1$. Let now G act continuously on a compact space X . Choose an arbitrary $x_0 \in X$. The push-forward, ν , of the measure μ to X along the corresponding orbit map, given by $\nu(B) = \mu\{g \in G : gx_0 \in B\}$, is again a finitely-additive Borel measure on X of total mass one, invariant under translations by \underline{G} and having the same “blowing-up” property: if $\nu(B) > 0$ and V is a non-empty open subset of G , then $\nu(VB) = 1$. Given a finite cover γ of X , an element W of the unique uniformity on X , and a finite subset F of \underline{G} , there is at least one $A \in \gamma$ with $\nu(A) > 0$, consequently $\nu(W[A]) = 1$ and for all $g \in F$ the translates $g \cdot W[A]$, having full measure each, must overlap. This can be used to construct a Cauchy filter \mathcal{F} of closed subsets of X with $A \in \mathcal{F}$, $g \in \underline{G}$ implying $gA \in \mathcal{F}$. The only point of $\cap \mathcal{F}$ is fixed under the action of \underline{G} and therefore of G as well. \square

For instance, the groups in Examples 1 and 2 are Lévy groups, and so is the isometry group $\text{Iso}(\mathbb{U})$ with the Polish topology [46].

Theorem of Gromov and Milman cannot be inverted, because the extremely amenable groups from Example 3 are not Lévy: they simply do not contain any non-trivial compact subgroups. What if such subgroups are present? The following is a reasonable general reading of an old question by Furstenberg discussed at the end of [31].

Question 17. *Suppose G is an extremely amenable topological group containing a net of compact subgroups (K_α) whose union is everywhere dense in G . Is G a Lévy group?*³ 1017 ?

Question 18. *Provided the answer is yes, is the family (K_α) a Lévy family?*⁴ 1018 ?

A candidate for a “natural” counter-example is the group $SU(\infty)$, the inductive limit of the family of special unitary groups of finite rank embedded one into the other via $SU(n) \ni V \mapsto \begin{pmatrix} \mathbb{I} & 0 \\ 0 & V \end{pmatrix} \in SU(n+1)$. Equip $SU(\infty)$ with the inductive limit topology, that is, the finest topology inducing the given topology on each $SU(n)$.

Question 19. *Is the group $SU(\infty)$ with the inductive limit topology extremely amenable?* 1019 ?

If the answer is yes, then Questions 17 and 4 are both answered in the negative.

Historically the first example of an extremely amenable group was constructed by Herer and Christensen [34]. Theirs was an abelian topological group without strongly continuous unitary representations in Hilbert spaces (an *exotic group*).

³I. Farah and S. Solecki have announced a counter-example (May 2006).

⁴Cf. the previous footnote.

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Question 20. *Is the exotic group constructed in [34] a Lévy group?*

The following result shows that the properties of Lévy groups are diametrically opposed to those of locally compact groups in the setting of ergodic theory as well as topological dynamics.

Theorem (Glasner–Tsirelson–Weiss [25]). *Let a Polish Lévy group act in a Borel measurable way on a Polish space X . Let μ be a Borel probability measure on X invariant under the action of G . Then μ is supported on the set of G -fixed points.*

? 1021 **Question 21** (Glasner–Tsirelson–Weiss, *ibid.*). *Is the same conclusion true if one only assumes that the measure μ is quasi-invariant under the action of G , that is, for all $g \in G$ and every null-set $A \subseteq X$, the set gA is null?*

Recall that a compact G -space X is called *minimal* if the orbit of every point is everywhere dense in X . To every topological group G there is associated the *universal minimal flow*, $\mathcal{M}(G)$, which is a minimal compact G -space uniquely determined by the property that every other minimal G -space is an image of $\mathcal{M}(G)$ under an equivariant continuous surjection. (See [3].) For example, G is extremely amenable if and only if $\mathcal{M}(G)$ is a singleton. If G is compact, then $\mathcal{M}(G) = G$, but for locally compact non-compact groups, starting with \mathbb{Z} , the flow $\mathcal{M}(G)$ is typically very complicated and highly non-constructive, in particular it is never metrizable [37]. A discovery of the recent years has been that even non-trivial universal minimal flows of “large” topological groups are sometimes manageable.

Example 4. The flow $\mathcal{M}(\text{Homeo}_+(\mathbb{S}^1))$ is the circle \mathbb{S}^1 itself, equipped with the canonical action of the group $\text{Homeo}_+(\mathbb{S}^1)$ of orientation-preserving homeomorphisms, with the compact-open topology [42].

Example 5. Let S_∞ denote the infinite symmetric group, that is, the Polish group of all bijections of the countably infinite discrete space ω onto itself, equipped with the topology of pointwise convergence. The flow $\mathcal{M}(S_\infty)$ can be identified with the set of all linear orders on ω with the topology induced from $\{0, 1\}^{\omega \times \omega}$ under the identification of each order with the characteristic function of the corresponding relation [26].

Example 6. Let $C = \{0, 1\}^\omega$ stand for the Cantor set. The minimal flow $\mathcal{M}(\text{Homeo}(C))$ can be identified with the space of all maximal chains of closed subsets of C , equipped with the Vietoris topology. This is the result of Glasner and Weiss [27], while the space of maximal chains was introduced into the dynamical context by Uspenskij [59].

? 1022 **Question 22** (Uspenskij). *Give an explicit description of the universal minimal flow of the homeomorphism group $\text{Homeo}(X)$ of a closed compact manifold X in dimension $\dim X > 1$ (with the compact-open topology).*

? 1023 **Question 23** (Uspenskij). *The same question for the group of homeomorphisms of the Hilbert cube $Q = \mathbb{I}^\omega$.*

Note that both X and Q form minimal flows for the respective homeomorphism groups, but they are not universal [59]. Interesting recent advances on both questions 22 and 23 belong to Yonatan Gutman [32].

Question 24 (Uspenskij). *Is the pseudoarc P the universal minimal flow for its own homeomorphism group?* 1024 ?

A recent investigation [36] might provide means to attack this problem.

Let G be a topological group. The completion of G with regard to the left uniform structure (the *left* completion), denoted by \hat{G}^L , is a topological semigroup with jointly continuous multiplication [49, Prop. 10.2(a)], but in general not a topological group [13]. Note that every left uniformly continuous real-valued function f on G extends to a unique continuous function \hat{f} on \hat{G}^L . Say that such an f is *oscillation stable* if for every $\varepsilon > 0$ there is a right ideal J in the topological semigroup \hat{G}^L with the property that the values of \hat{f} at any two points of J differ by $< \varepsilon$. If H is a closed subgroup of G , say that the homogeneous space G/H is *oscillation stable* if every bounded left uniformly continuous function f on G that factors through the quotient map $G \rightarrow G/H$ is oscillation stable. If G/H is not oscillation stable, we say that G/H has *distortion*.

Example 7. The unit sphere \mathbb{S}^∞ in the separable Hilbert space ℓ^2 , considered as the homogeneous factor-space of the unitary group $U(\ell^2)$ with the strong topology, has distortion. It means that there exists a uniformly continuous function $f: \mathbb{S}^\infty \rightarrow \mathbb{R}$ whose range of values on the intersection of \mathbb{S}^∞ with every infinite-dimensional linear subspace contains the interval (say) $[0, 1]$. This is a famous and very difficult result by Odell and Schlumprecht [39], answering a 30 year-old problem. The following question is well-known in geometric functional analysis.

Question 25. *Does there exist a direct proof of Odell and Schlumprecht's result, based on the intrinsic geometry of the unit sphere and/or the unitary group?* 1025 ?

Example 8. The set $[\mathbb{Q}]^n$ of all n -subsets of \mathbb{Q} , considered as a homogeneous factor-space of $\text{Aut}(\mathbb{Q}, \leq)$, is oscillation stable if and only if $n = 1$. For $n = 1$, oscillation stability simply means that for every finite colouring of \mathbb{Q} , there is a monochromatic subset A order-isomorphic to \mathbb{Q} (this is obvious). For $n \geq 2$, distortion of $[\mathbb{Q}]^n$ means the existence of a finite colouring of this set with $k \geq 2$ colours such that for every subset A order-isomorphic to the rationals the set $[A]^n$ contains points of all k colours. This follows easily from classical Sierpiński's partition argument [53], cf. [45, Example 5.1.27].

The above setting for analysing distortion/oscillation stability in the context of topological transformation groups was proposed in [37] and discussed in [45]. The most substantial general result within this framework is presently the following.

Theorem (Hjorth [35]). *Let G be a Polish topological group. Considered as a G -space with regard to the action on itself by left translations, G has distortion whenever $G \neq \{e\}$.*

Question 26 (Hjorth [35]). *Let E be a separable Banach space and let \mathbb{S}_E denote the unit sphere of E viewed as an $\text{Iso}(E)$ -space, where the latter group is equipped with the strong operator topology. Is it true that the $\text{Iso}(E)$ -space \mathbb{S}_E has distortion?* **1026 ?**

Note of caution: this would not, in general, mean that E has distortion in the sense of theory of Banach spaces [7, Chapter 13], as the two concepts only coincide for Hilbert spaces.

For an ultrahomogeneous separable metric space X , oscillation stability of X equipped with the standard action of the Polish group of isometries $\text{Iso}(X)$ is equivalent to the following property. For every finite cover γ of X , there is an $A \in \gamma$ such that for each $\varepsilon > 0$, the ε -neighbourhood of A contains an isometric copy of X . The following could provide a helpful insight into question 25.

? 1027 Question 27. *Is the metric space \mathbb{U}_1 oscillation stable?*

The Urysohn metric space \mathbb{U} itself has distortion, but for trivial reasons, just like any other unbounded connected ultrahomogeneous metric space.

The oscillation stability of a metric space X whose distance assumes a discrete collection of values is equivalent to the property that whenever X is partitioned into two subsets, at least one of them contains an isometric copy of X . The Urysohn metric space $\mathbb{U}_{\{0,1,2\}}$ universal for the class of metric spaces whose distances take values 0, 1, 2 is oscillation stable, because it is isometric to the path metric space associated to the *infinite random graph* R , and oscillation stability is an immediate consequence of an easily proved property of R known as *indestructibility* (cf. [8]). Very recently, Delhomme, Laflamme, Pouzet, and Sauer [12] have established oscillation stability of the universal Urysohn metric space $\mathbb{U}_{\{0,1,2,3\}}$ with the distance taking values 0, 1, 2, 3. The following remains unknown.

? 1028 Question 28. *Let $n \in \mathbb{N}$, $n \geq 4$. Is the universal Urysohn metric space $\mathbb{U}_{\{0,1,\dots,n\}}$ oscillation stable?*

Resolving the following old question may help.

? 1029 Question 29 (M. Fréchet [19], p. 100; P.S. Alexandroff [55]). *Find a model for the Urysohn space \mathbb{U} , that is, a concrete realization.*

Several such models are known for the random graph (thence, $\mathbb{U}_{\{0,1,2\}}$), cf. [8].

? 1030 Question 30. *Find a model for the metric space $\mathbb{U}_{\{0,1,2,3\}}$.*

In connection with Uspenskij's examples of universal second-countable topological groups [56, 57], including $\text{Iso}(\mathbb{U})$, the following remains unresolved.

? 1031 Question 31 (V.V. Uspenskij [58]). *Does there exist a universal topological group of every given infinite weight τ ?*

? 1032 Question 32 (V.V. Uspenskij). *The same, for any uncountable weight?*

? 1033 Question 33 (A.S. Kechris). *Does there exist a co-universal Polish topological group G , that is, such that every other Polish group is a topological factor-group of G ?*

In the abelian case, the answer is in the positive [52].

Question 34 (A.S. Kechris). *Is every Polish topological group a topological factor-group of a subgroup of $U(\ell^2)$ with the strong topology?* **1034 ?**

Again, in the abelian case the answer is in the positive [22].

Question 35. *Is the free topological group $F(X)$ [28] on a metrizable compact space isomorphic to a topological subgroup of the unitary group $U(\mathcal{H})$ of a suitable Hilbert space \mathcal{H} , equipped with the strong topology?* **1035 ?**

Galindo has announced [20] a positive answer for free abelian topological groups. Uspenskij [60] has given a very elegant proof of a more general result: the free abelian topological group $A(X)$ of a Tychonoff space embeds into $U(\mathcal{H})$ as a topological subgroup. This suggests a more general version of the same question:

Question 36. *The same question for an arbitrary Tychonoff space X .* **1036 ?**

In connection with questions 34, 35 and 36, let us remind the following old problem.

Question 37 (A.I. Shtern [51]). *What is the intrinsic characterization of topological subgroups of $U(\ell^2)$ (with the strong topology)?* **1037 ?**

A unitary representation π of a topological group G in a Hilbert space \mathcal{H} (that is, a strongly continuous homomorphism $G \rightarrow U(\mathcal{H})$) *almost has invariant vectors* if for every compact $F \subseteq G$ and every $\varepsilon > 0$ there is a $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ and $\|\pi_g \xi - \xi\| < \varepsilon$ for every $g \in F$. A topological group G has *Kazhdan's property (T)* if, whenever a unitary representation of G almost has invariant vectors, it has an invariant vector of norm one. For an excellent account of this rich theory, see the book [33] and especially its many times extended and updated English version, currently in preparation and available on-line [6].

Most of the theory is concentrated in the locally compact case. Bekka has shown in [5] that the group $U(\ell^2)$ with the strong topology has property (T).

Question 38 (Bekka [5]). *Does the group $U(\ell^2)$ with the uniform topology have property (T)?* **1038 ?**

Question 39 (Bekka [5]). *Does the unitary group $U(\ell^2(\Gamma))$ of a non-separable Hilbert space $(\Gamma > \aleph_0)$, equipped with the strong topology, have property (T)?* **1039 ?**

Here is a remarkable “large” topological group that has been receiving much attention recently. Let $\|\cdot\|_2$ denote the Hilbert-Schmidt norm on the $n \times n$ matrices, $\|A\|_2 = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}$, and let d_n be the *normalized* Hilbert-Schmidt metric on the unitary group $U(n)$, that is, $d_n(u, v) = \frac{1}{\sqrt{n}} \|u - v\|_2$. Choose a free ultrafilter ξ on the natural numbers and denote by $U(\xi)_2$ the factor-group of the direct product $\prod_{n \in \mathbb{N}} U(n)$ by the normal subgroup $N_\xi = \{(x_n) : \lim_{n \rightarrow \xi} d_n(e, x_n) = 0\}$.

The following question is a particular case of Connes' Embedding Conjecture [10], for a thorough discussion see [40] and references therein.

Question 40 (Connes' Embedding Conjecture for Groups). *Is every countable group isomorphic to a subgroup of $U(\xi)_2$ (as an abstract group)?* 1040 ?

Groups isomorphic to subgroups of $U(\xi)_2$ are called *hyperlinear*. Here are some of the most important particular cases of the above problem.

? 1041–1043 **Question 41.** *Are countable groups from the following classes hyperlinear: (a) one-relator groups; (b) hyperbolic groups [29]; (c) groups amenable at infinity (a.k.a. topologically amenable groups, exact groups) [2]?*

Under the natural bi-invariant metric $d(x, y) = \lim_{n \rightarrow \xi} d_n(x_n, y_n)$, the group $U(\xi)_2$ is a complete non-separable metric group whose left and right uniformities coincide, isomorphic to a topological subgroup of $U(\ell^2(\mathfrak{c}))$ with the strong topology. Understanding the topological group structure of $U(\xi)_2$ may prove important.

The Connes' Embedding Conjecture itself can be reformulated in the language of topological groups as follows. Say, following [47], that a topological group G has *Kirchberg's property* if, whenever A and B are finite subsets of G with the property that every element of A commutes with every element of B , there exist finite subsets A' and B' of G that are arbitrarily close to A and B , respectively, such that every element of A' commutes with every element of B' , and the subgroups of G generated by A' and B' are relatively compact. As noted in [47], the deep results of [38], modulo a criterion from [17], immediately imply that the Connes Embedding Conjecture is equivalent to the statement that the unitary group $U(\ell^2)$ with the strong topology has Kirchberg's property.

? 1044–1045 **Question 42.** *Do the following topological groups have Kirchberg's property: (a) the infinite symmetric group S_∞ , (b) the group $\text{Aut}(X, \mu)$ of measure-preserving transformations of a standard Lebesgue measure space with the coarse topology?*

It was shown in [47] that $\text{Iso}(\mathbb{U})$ has Kirchberg's property.

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